FIXED POINT SETS OF S^1-ACTIONS ON THE SPACES WHOSE RATIONAL COHOMOLOGY RINGS ARE EVENLY GRADED

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1. Introduction

Let $G = S^1$ be the circle group, and X a connected finite G-CW-complex whose rational cohomology ring is evenly graded; that is

(1.1) $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \cdots, x_n]/(\varphi_1, \cdots, \varphi_m)$,

where $\deg x_i = 2k_i \geqq 2$ $(1 \leqq i \leqq n)$ and φ_i are homogeneous elements. We establish a method of determining the possibilities of the rational cohomology type of the fixed point set of G on X (Theorem 3.7). The method is an application of that originated and improved by K. Hokama in [2] and [4] respectively. Combining Theorem 3.7 with a result of V. Puppe in [5] (Theorem 3.8), the problem of existence for connected finite G-CW-complex whose rational cohomology ring is evenly graded is reduced to an algebraic one. We apply the result to three cases (Theorems 4.1, 4.2 and 4.3). A result of G. E. Bredon [1] applied to the case

$$X \sim_{\mathbb{Q}} S^{2m} \times S^{2n}$$

is improved slightly (Theorem 4.1). In the final section, we construct G-CW-complexes which give examples in Theorems 4.2 and 4.3 except for the case (4).

2. Preliminaries

Let R (resp. ${}^aR$) be the polynomial ring $\mathbb{Q}[t,x_1,\cdots,x_n]$ (resp. $\mathbb{Q}[x_1,\cdots,x_n]$), where $\deg t = 2$, $\deg x_i = 2k_i$ $(1 \leqq i \leqq n)$ and $1 \leqq k_1 \leqq \cdots \leqq k_n$. Let ${}^a\colon R \longrightarrow {}^aR$ denotes the ring homomorphism defined by

$$ {}^aF(x_1,\cdots,x_n) = F(1,x_1,\cdots,x_n) $$

for every $F \in R$, and ${}^h\colon {}^aR\setminus\{0\} \longrightarrow R$ a map defined by

$$ {}^hf(t,x_1,\cdots,x_n) = t^{\delta(f)/2}f(x_1/t^{k_1},\cdots,x_n/t^{k_n}) $$

for every $f \in {}^aR\setminus\{0\}$, where $\delta(f)$ denotes the total degree of f. For any ideal J in R, we set ${}^aJ = \{{}^aF \mid F \in J\}$. Let $\varphi_i \in {}^aR$ and $f_i \in R$ be homogeneous elements $(1 \leqq i \leqq m)$, and suppose

(I) $f_i(0,x_1,\cdots,x_n) = \varphi_i(x_1,\cdots,x_n)$ $\quad(1 \leqq i \leqq m)$.

(II) $\dim_{\mathbb{Q}} {}^aR/(\varphi_1,\cdots,\varphi_m) < \infty$.

Assume that ${}^aR/(\varphi_1,\cdots,\varphi_m)$ has a basis $M = \{[y_i] \mid 1 \leqq i \leqq h\}$ over $\mathbb{Q}$, where y_i is a homogeneous element $(1 \leqq i \leqq h)$, $y_1 = 1$ and $0 = \deg y_1 \leqq \cdots \leqq \deg y_h = 2N \geqq 2k_n$. Then we have the following lemma.

Lemma 2.1. (1) The $\mathbb{Q}[t]$-module $R/(f_1,\cdots,f_m)$ is generated by M.

(2) The $\mathbb{Q}$-module ${}^aR/{}^a(f_1,\cdots,f_m)$ is generated by M.

(3) The following conditions are equivalent:

i) If $tf \in (f_1,\cdots,f_m)$, then $f \in (f_1,\cdots,f_m)$,

ii) $R/(f_1,\cdots,f_m)$ is a free $\mathbb{Q}[t]$-module with a basis M,

iii) $\dim_{\mathbb{Q}} {}^aR/{}^a(f_1,\cdots,f_m) = \dim_{\mathbb{Q}} {}^aR/(\varphi_1,\cdots,\varphi_m) = h$.

Proof. (1) Let $f(t,x) \in R$ be a homogeneous element of positive degree. Then, by the assumption

$$f(0,x) = \sum_{i=1}^{h} a_i y_i + \sum_{j=1}^{m} k_j(x)\varphi_j(x)$$

for some $a_i \in \mathbb{Q}$ $(1 \leqq i \leqq h)$ and $k_j(x) \in \mathbb{Q}[x]$ $(1 \leqq j \leqq m)$. This implies that

$$f(t,x) - \sum_{i=1}^{h} a_i y_i - \sum_{j=1}^{m} k_j(x) f_j(t,x) = tg(t,x)$$

for a homogeneous element $g(t,x) \in R$. It is shown by the induction with respect to $\deg f(t,x)$ that

$$g(t,x) = \sum_{i=1}^{h} h_i(t) y_i + \sum_{j=1}^{m} g_j(t,x) f_j(t,x)$$

for some $h_i(t) \in \mathbb{Q}[t]$ $(1 \leqq i \leqq h)$ and $g_j(t,x) \in R$ $(1 \leqq j \leqq m)$. Then we have

$$f(t,x) = \sum_{i=1}^{h} (a_i + th_i(t)) y_i + \sum_{j=1}^{m} (k_j(x) + tg_j(t,x)) f_j(t,x).$$

(2) Let $f(x) \in {}^aR \setminus \{0\}$ be a polynomial. It follows from (1) that we have

$${}^hf(t,x) = \sum_{i=1}^{h} h_i(t) y_i + \sum_{j=1}^{m} k_j(t,x) f_j(t,x)$$

for some $h_i(t) \in \mathbb{Q}[t]$ $(1 \leqq i \leqq h)$ and $k_j(t,x) \in R$ $(1 \leqq j \leqq m)$. Then we have

$$f(x) = \sum_{i=1}^{h} h_i(1) y_i + \sum_{j=1}^{m} k_j(1,x)\, {}^af_j(x).$$

(3) Suppose i) and $\sum_{i=1}^{h} h_i(t) y_i \in (f_1, \cdots, f_m)$. Then

$$\sum_{i=1}^{h} h_i(0) y_i \in (\varphi_1, \cdots, \varphi_m),$$

and hence $h_i(0) = 0$ $(1 \leqq i \leqq h)$. Suppose that $h_i(t) = tg_i(t)$ $(1 \leqq i \leqq h)$. Then

$$t\sum_{i=1}^{h} g_i(t) y_i = \sum_{i=1}^{h} h_i(t) y_i \in (f_1, \cdots, f_m),$$

and hence $\sum_{i=1}^{h} g_i(t) y_i \in (f_1, \cdots, f_m)$. It is shown by the induction with respect to the degree of $\sum_{i=1}^{h} h_i(t) y_i$ that $g_i(t) = 0$ $(1 \leqq i \leqq h)$. Then $h_i(t) = 0$ $(1 \leqq i \leqq h)$. Thus i) implies ii).

Suppose that $f(t,x) \in R$ be a homogeneous element such that $tf(t,x) \in (f_1,\cdots,f_m)$ and $f(t,x)$ is not contained in $(f_1,\cdots,f_m)$. By the assumption,

$$f(0,x) = \textstyle\sum_{i=1}^{h} a_i y_i + \sum_{j=1}^{m} k_j(x)\varphi_j(x)$$

for some $a_i \in \mathbb{Q}$ $(1 \leqq i \leqq h)$ and $k_j(x) \in \mathbb{Q}[x]$ $(1 \leqq j \leqq m)$. If $a_i = 0$ $(1 \leqq i \leqq h)$, then

$$f(t,x) - \textstyle\sum_{j=1}^{m} k_j(x) f_j(t,x) = tg(t,x)$$

for some $g(t,x) \in R$. Then $g(t,x)$ is not contained in $(f_1,\cdots,f_m)$ and $t^2 g(t,x) \in (f_1,\cdots,f_m)$. Thus there exists $f(t,x) \in R$ and a positive integer N_1 such that $t^{N_1} f(t,x) \in (f_1,\cdots,f_m)$ and $[f(0,x)]$ has non-zero component with respect to the basis M. Suppose that

$$f(0,x) - \textstyle\sum_{i=1}^{k} a_i y_i \in (\varphi_1,\cdots,\varphi_m)$$

for some integer k with $1 \leqq k \leqq h$ and $a_i \in \mathbb{Q}$ $(1 \leqq i \leqq k)$ with $a_k \neq 0$. Then $R/(f_1,\cdots,f_m,f)$ is generated by $M\setminus\{[y_k]\}$ over $\mathbb{Q}[t]$, and ${}^{a}R/{}^{a}(f_1,\cdots,f_m,f)$ is generated by $M\setminus\{[y_k]\}$ over $\mathbb{Q}$. Since

$${}^{a}(f_1,\cdots,f_m,f) = {}^{a}(f_1,\cdots,f_m,t^{N_1}f) = {}^{a}(f_1,\cdots,f_m),$$

this implies that $\dim_{\mathbb{Q}} {}^{a}R/{}^{a}(f_1,\cdots,f_m) \leqq h-1$. Hence iii) does not hold. Thus iii) implies i).

Suppose that $\sum_{i=1}^{k} a_i y_i \in {}^{a}(f_1,\cdots,f_m)$ for some k with $1 \leqq k \leqq h$ and $a_i \in \mathbb{Q}$ $(1 \leqq i \leqq k)$ with $a_k \neq 0$. Then

$$\textstyle\sum_{i=1}^{k} a_i t^{N_1-(\deg y_i)/2} y_i \in (f_1,\cdots,f_m)$$

for some integer $N_1 \geqq \deg y_k$. Hence ii) does not holds. Thus ii) implies iii). This completes the proof of (3).

q.e.d.

Let $\{(c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) \mid 1 \leqq \alpha \leqq k\}$ be a set of rational zero points of the ideal ${}^a(f_1,\cdots,f_m) \subset {}^aR$; that is, $c_i^{(\alpha)} \in \mathbb{Q}$ and $f_i(1,c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) = 0$ $(1 \leqq i \leqq m,\ 1 \leqq \alpha \leqq k)$. For $1 \leqq i \leqq n$ and $1 < j \leqq k_i$, $g_{i,j} \in R$ denotes either 0 or a homogeneous element of degree $2N_0$ with $N_0 \geqq N$ and $g_{i,j}(1,c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) = 0$ $(1 \leqq \alpha \leqq k)$. Let S (resp. ${}^aS$) be the polynomial ring

$$\mathbb{Q}[\{t\} \cup \{x_{i,j} \mid 1 \leqq i \leqq n,\ 1 \leqq j \leqq k_i\}]$$

(resp. $\mathbb{Q}[\{x_{i,j} \mid 1 \leqq i \leqq n,\ 1 \leqq j \leqq k_i\}]$), where $\deg x_{i,j} = 2j$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq k_i)$ and $\deg t = 2$. Consider homomorphisms $J_\alpha\colon R \longrightarrow S$ (resp. ${}^aJ_\alpha\colon {}^aR \longrightarrow {}^aS$) defined by

$$J_\alpha(f) = f(t,c_1^{(\alpha)}t^{k_1}+\textstyle\sum_{j=1}^{k_1}x_{1,j}t^{k_1-j},\cdots,c_n^{(\alpha)}t^{k_n}+\sum_{j=1}^{k_n}x_{n,j}t^{k_n-j})$$

(resp. ${}^aJ_\alpha(f) = f(c_1^{(\alpha)}+\sum_{j=1}^{k_1}x_{1,j},\cdots,c_n^{(\alpha)}+\sum_{j=1}^{k_n}x_{n,j})$) $(1 \leqq \alpha \leqq k)$. Denote by I_α the ideal generated in ${}^aS$ by the coefficients of $J_\alpha(f_i)$ $(1 \leqq i \leqq m)$ and $J_\alpha(g_{i,j})-x_{i,j}t^{N_0-j}$ $(1 \leqq i \leqq n,\ 1 < j \leqq k_i)$ with respect to t. Set $q_\alpha = J_\alpha^{-1}(I_\alpha S)$ $(1 \leqq \alpha \leqq k)$. Consider the induced homomorphisms

$$j_\alpha\colon R/(f_1,\cdots,f_m) \longrightarrow S/I_\alpha S$$

and ${}^aj_\alpha\colon {}^aR/{}^a(f_1,\cdots,f_m) \longrightarrow {}^aS/I_\alpha$ $(1 \leqq \alpha \leqq k)$.

Lemma 2.2. Let α be an integer with $1 \leqq \alpha \leqq k$.

(1) The graded group ${}^aS/I_\alpha$ equals to zero in the degrees $> 2N$.

2) The ideal q_α is primary with the radical

$$\sqrt{q_\alpha} = (x_1-c_1^{(\alpha)}t^{k_1},\cdots,x_n-c_n^{(\alpha)}t^{k_n}).$$

(3) The ideal ${}^aq_\alpha$ coincides with ${}^aJ_\alpha^{-1}(I_\alpha)$, and ${}^aj_\alpha$ induces an isomorphism ${}^a\tilde{j}_\alpha\colon {}^aR/{}^aq_\alpha \xrightarrow{\cong} {}^aS/I_\alpha$.

Proof. (1) Let $g(x_{i,j}) \in {}^aS$ be a homogeneous element. Set $f = g(g_{i,j}t^{N_0(j-1)}) \in R$, where

$$g_{i,1} = x_i t^{N_0-k_i} - c_i^{(\alpha)} t^{N_0} - \sum_{j=2}^{k_i} g_{i,j} \qquad (1 \leqq i \leqq n).$$

It follows from the definition that $J_\alpha(f)$ is congruent to $gt^{(N_0-1)(\deg g)/2}$ (mod $I_\alpha S$). By Lemma 2.1 (1) there exist $h_i(t) \in \mathbb{Q}[t]$ $(1 \leqq i \leqq h)$ and $k_i(t,x) \in R$ $(1 \leqq i \leqq m)$ with

$$f = \sum_{i=1}^h h_i(t)y_i + \sum_{i=1}^m k_i(t,x)f_i(t,x).$$

Then $gt^{(N_0-1)(\deg g)/2}$ is congruent to $\sum_{i=1}^h h_i(t)J_\alpha(y_i)$ (mod $I_\alpha S$), the degree with respect to $\{x_{i,j}\}$ of which is at most $2N$. If $\deg g > 2N$, then $g \in I_\alpha$.

(2) It follows from (1) that $\sqrt{I_\alpha} = (x_{i,j})$. This implies that $\sqrt{I_\alpha S} = (x_{i,j})$, and $I_\alpha S$ is a primary ideal. Since q_α is the inverse image of $I_\alpha S$ by a ring homomorphism J_α, we obtain (2).

(3) Suppose that $f \in {}^aq_\alpha$. Choose $F \in q_\alpha$ with ${}^aF = f$. Then, the coefficients of $J_\alpha(F)$ with respect to t are contained in I_α, and the sum of which is equal to ${}^aJ_\alpha(f) = {}^a(J_\alpha(F))$. This implies that $f \in {}^aJ_\alpha^{-1}(I_\alpha)$. Conversely, suppose that $f \in {}^aJ_\alpha^{-1}(I_\alpha)$ and $f \neq 0$. Then ${}^a(J_\alpha({}^hf)) = {}^aJ_\alpha(f) \in I_\alpha$. Since I_α is a homogeneous ideal and the set of the coefficients of $J_\alpha({}^hf)$ with respect to t coincides to that of the homogeneous components of ${}^aJ_\alpha(f)$, we have $J_\alpha({}^hf) \in I_\alpha S$. This implies that ${}^hf \in q_\alpha$, and $f = {}^a({}^hf) \in {}^aq_\alpha$. Thus we obtain the first part of (3) and a monomorphism ${}^a\tilde{j}_\alpha : {}^aR/{}^aq_\alpha \longrightarrow {}^aS/I_\alpha$. It follows from the proof of (1) that ${}^a\tilde{j}_\alpha$ is also an epimorphism. This completes the proof of (3). q.e.d.

In order to state the next lemma, we set

$$j = \oplus_{\alpha=1}^{k} j_\alpha : R/(f_1,\cdots,f_m) \longrightarrow \oplus_{\alpha=1}^{k} S/I_\alpha S$$

and $\quad {}^a j = \oplus_{\alpha=1}^{k} {}^a j_\alpha : {}^a R/{}^a(f_1,\cdots,f_m) \longrightarrow \oplus_{\alpha=1}^{k} {}^a S/I_\alpha .$

Lemma 2.3. (1) j is surjective in the degrees $\geqq 2N$.

(2) ${}^a j$ is an epimorphism.

(3) $\sum_{\alpha=1}^{k} \dim_{\mathbb{Q}} {}^a S/I_\alpha \leqq h$.

(4) $\sum_{\alpha=1}^{k} \dim_{\mathbb{Q}} {}^a R/{}^a q_\alpha \leqq h$.

(5) $(f_1,\cdots,f_m) \subset \cap_{\alpha=1}^{k} q_\alpha$.

(6) ${}^a(f_1,\cdots,f_m) \subset \cap_{\alpha=1}^{k} {}^a q_\alpha$.

(7) The following conditions are equivalent:

i) j is a monomorphism,

ii) ${}^a j$ is an isomorphism,

iii) $\sum_{\alpha=1}^{k} \dim_{\mathbb{Q}} {}^a S/I_\alpha = h$,

iv) $\sum_{\alpha=1}^{k} \dim_{\mathbb{Q}} {}^a R/{}^a q_\alpha = h$,

v) $(f_1,\cdots,f_m) = \cap_{\alpha=1}^{k} q_\alpha$,

vi) ${}^a(f_1,\cdots,f_m) = \cap_{\alpha=1}^{k} {}^a q_\alpha$.

If this is the case, the set of zero points of the ideal ${}^a(f_1,\cdots,f_m) \subset {}^a R$ coincides with $\{(c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) \mid 1 \leqq \alpha \leqq k\}$.

Proof. (1) We show that for each α $(1 \leqq \alpha \leqq k)$ and homogeneous element $g \in {}^a S$, there exist a homogeneous element $f \in R$ and an integer N_1 with $0 \leqq N_1 \leqq \max\{0, N-(\deg g)/2\}$, $J_\beta(f) \in I_\beta S$ $(\beta \neq \alpha)$ and $J_\alpha(f)-gt^{N_1} \in I_\alpha S$. If $\deg g > 2N$, then $f = 0$ and $N_1 = 0$ satisfies the above condition. Let $\deg g = 2\ell \leqq 2N$, and set

$$F_1 = g(g_{i,j}t^{N_0(j-1)})\cdot\Pi_{\beta\neq\alpha}(x_{i(\beta)}-c_{i(\beta)}^{(\beta)}t^{k_{i(\beta)}})^{N+1},$$

where $g_{i,1} = x_i t^{N_0-k_i} - c_i^{(\alpha)}t^{N_0} - \sum_{j=2}^{k_i} g_{i,j}$ $(1 \leqq i \leqq n)$ and $i(\beta)$ is an integer with $c_{i(\beta)}^{(\beta)} \neq c_{i(\beta)}^{(\alpha)}$ $(\beta \neq \alpha)$. Then we have $J_\beta(F_1) \in I_\beta S$ $(\beta \neq \alpha)$ and

$$J_\alpha(F_1) \equiv ct^{\ell(N_0-1)+N(\alpha)}g + \sum_{i=0}^{N(\alpha)-1} t^{\ell(N_0-1)+i} g_i \pmod{I_\alpha S},$$

where $N(\alpha) = \sum_{\beta\neq\alpha}(N+1)k_{i(\beta)}$, $0 \neq c \in \mathbb{Q}$, $g_i \in {}^aS$ $(0 \leqq i < N(\alpha))$ and $\deg g_i > 2\ell$ if $g_i \neq 0$. Set $N_2 = \max\{N, \ell N_0+N(\alpha)\}$. It follows from the inductive hypothesis that there exists a homogeneous element $F \in R$ of degree $2N_2$ with $J_\beta(F) \in I_\beta S$ $(\beta \neq \alpha)$ and $J_\alpha(F)-gt^{N_2-\ell} \in I_\alpha S$. By Lemma 2.1 (1) there exist $h_i(t) \in \mathbb{Q}[t]$ $(1 \leqq i \leqq h)$ and $k_i(t,x) \in R$ $(1 \leqq i \leqq m)$ with

$$F = \sum_{i=1}^h h_i(t)y_i + \sum_{i=1}^m k_i(t,x)f_i(t,x).$$

Set $\sum_{i=1}^h h_i(t)y_i = ft^{N_2-N}$, where $f \in R$ is a homogeneous element of degree $2N$. Then, we have $J_\beta(f) \in I_\beta S$ $(\beta \neq \alpha)$ and $J_\alpha(f)-gt^{N-\ell} \in I_\alpha S$. Thus (1) is proved by the induction with respect to $(\deg g)/2$.

(2) In the proof above, ${}^aj([{}^af]) = [g]$. This implies (2).

(3) It follows from (2) and Lemma 2.1 (2) that we have

$$\sum_{\alpha=1}^k \dim_{\mathbb{Q}} {}^aS/I_\alpha \leqq \dim_{\mathbb{Q}} {}^aR/{}^a(f_1,\cdots,f_m) \leqq h.$$

(4) is a direct consequence of (3) and Lemma 2.2 (3).

(5) and (6) are immediate from the definition.

(7) It is evident that i) is equivalent to v), ii) is equivalent to vi), iii) is equivalent to iv) and v) is equivalent to vi).

Suppose i). Then $R/(f_1,\cdots,f_m)$ is a torsionfree

$\mathbb{Q}[t]$-module ($\oplus_{\alpha=1}^{k} S/I_\alpha S$ is a free $\mathbb{Q}[t]$-module). It follows from Lemma 2.1 (3) and ii) that

$$h = \dim_{\mathbb{Q}} {}^aR/{}^a(f_1,\cdots,f_m) = \textstyle\sum_{\alpha=1}^{k} \dim_{\mathbb{Q}} {}^aS/I_\alpha .$$

Thus i) implies iii).

Suppose iii). It follows from (2) and Lemma 2.1 (2) that

$$h \geqq \dim_{\mathbb{Q}} {}^aR/{}^a(f_1,\cdots,f_m) \geqq \textstyle\sum_{\alpha=1}^{k} \dim_{\mathbb{Q}} {}^aS/I_\alpha = h.$$

This implies that $h = \dim_{\mathbb{Q}} {}^aR/{}^a(f_1,\cdots,f_m)$ and ${}^aj$ is an isomorphism. Thus iii) implies ii). This completes the proof of (7). q.e.d.

3. Equivariant cohomology rings

Let X be a connected finite G-CW-complex whose rational cohomology ring is evenly graded; that is, there exists an isomorphism

(3.1) $i(X)\colon \mathbb{Q}[x_1,\cdots,x_n]/(\varphi_1,\cdots,\varphi_m) \longrightarrow H^*(X;\mathbb{Q})$,

where $\deg x_i = 2k_i \geqq 2$ $(1 \leqq i \leqq n)$ and φ_i is a homogeneous element $(1 \leqq i \leqq m)$. Let $\tau\colon EG \times_G \mathbb{C} \longrightarrow BG$ be the complex line bundle associated to a universal G-bundle $EG \longrightarrow BG$. Then we have

(3.2) $H^*(BG;\mathbb{Q}) \cong \mathbb{Q}[t]$, where $t \in H^2(BG;\mathbb{Q})$ is the Euler class of τ.

Let $\pi\colon X_G = EG \times_G X \longrightarrow BG$ be the associated bundle with fiber X, and $i\colon X \longrightarrow X_G$ the inclusion of a fiber. The equivariant cohomology ring of X is defined by $H_G^*(X;\mathbb{Q})$

$= H^*(X_G; \mathbb{Q})$. The induced homomorphism $\pi^*: H^*(BG; \mathbb{Q}) \longrightarrow H_G^*(X; \mathbb{Q})$ gives a $\mathbb{Q}[t]$-algebra structure to $H_G^*(X; \mathbb{Q})$. Then $H_G^{odd}(X; \mathbb{Q}) \cong 0$ and the sequence

$$(3.3) \quad 0 \longrightarrow H_G^{2i}(X; \mathbb{Q}) \xrightarrow{t} H_G^{2i+2}(X; \mathbb{Q}) \xrightarrow{i^*} H^{2i+2}(X; \mathbb{Q}) \longrightarrow 0$$

is exact. Choose $a_i \in H_G^{2k_i}(X; \mathbb{Q})$ with $i^*(a_i) = i(X)([x_i])$ $(1 \leqq i \leqq n)$. Then $H_G^*(X; \mathbb{Q})$ is generated by $\{a_i \mid 1 \leqq i \leqq n\}$ as a $\mathbb{Q}[t]$-algebra. Let $I(X_G): R \longrightarrow H_G^*(X; \mathbb{Q})$ be the $\mathbb{Q}[t]$-algebra homomorphism defined by setting $I(X_G)(x_i) = a_i$ $(1 \leqq i \leqq n)$. Choose a homogeneous element $f_i \in R$ with $I(X_G)(f_i) = 0$ and $f_i(t, x_1, \cdots, x_n) = \varphi_i(x_1, \cdots, x_n)$ $(1 \leqq i \leqq m)$. The proof of following lemma is similar to that of [3, Lemma 2.2].

Lemma 3.4. The kernel of $I(X_G)$ coincides with the ideal $(f_1, \cdots, f_n)$ of R and $I(X_G)$ induces an isomorphism

$$i(X_G): R/(f_1, \cdots, f_m) \longrightarrow H_G^*(X; \mathbb{Q}).$$

Let $F = X^G$ be the fixed point set of the G-action on X with connected components $F_1, \cdots, F_k$. Let

$$j: F_G = BG \times F \hookrightarrow X_G$$

be the inclusion map. Then the sequence

$$(3.5) \quad 0 \longrightarrow H_G^*(X; \mathbb{Q}) \xrightarrow{j^*} H_G^*(F; \mathbb{Q}) \longrightarrow H^{*+1}(X/G, F; \mathbb{Q}) \longrightarrow 0$$

is exact. For $1 \leqq \alpha \leqq k$, let $p_\alpha: (F_\alpha)_G \longrightarrow F_\alpha$ be the projection to the second factor. Then we have an isomorphism

$$\iota_\alpha: \mathbb{Q}[t] \otimes H^*(F_\alpha; \mathbb{Q}) \longrightarrow H_G^*(F_\alpha; \mathbb{Q})$$

defined by setting $\iota_\alpha(h \otimes a) = hp_\alpha{}^*(a)$ for every $h \in \mathbb{Q}[t]$

and $a \in H^*(F_\alpha; \mathbb{Q})$. For $1 \leqq i \leqq n$, set

$$j^*(a_i) = \sum_{\alpha=1}^{k}(c_i^{(\alpha)} t^{k_i} + \sum_{j=1}^{k_i} t^{k_i - j} p_\alpha{}^*(a_{i,j}^{(\alpha)})),$$

where $a_{i,j}^{(\alpha)} \in H^{2j}(F_\alpha; \mathbb{Q})$ and $c_i^{(\alpha)} \in \mathbb{Q}$ $(1 \leqq \alpha \leqq k)$.

Lemma 3.6. (1) For $1 \leqq \alpha \leqq k$, $H^*(F_\alpha; \mathbb{Q})$ is generated by

$$\{a_{i,j}^{(\alpha)} | 1 \leqq i \leqq n,\ 1 \leqq j \leqq k_i\}.$$

(2) $(c_1^{(\alpha)}, \cdots, c_n^{(\alpha)}) \neq (c_1^{(\beta)}, \cdots, c_n^{(\beta)})$ if $\alpha \neq \beta$.

Proof. Suppose $a \in H^*(F_\alpha; \mathbb{Q})$. It follows from the exactness of the sequence (3.5) that there exist an integer $N_1 \geqq 0$ and an element $f \in R$ such that $j^*(I(X_G)(f)) = t^{N_1} p_\alpha{}^*(a)$. By the isomorphism ι_α, we see that there exist a polynomial $g \in {}^aS$ such that $g(a_{i,j}^{(\alpha)}) = a$. This completes the proof of (1).

In the proof above, set $a = 1$. Then we have

$$f(1, c_1^{(\alpha)}, \cdots, c_n^{(\alpha)}) = 1$$

and $f(1, c_1^{(\beta)}, \cdots, c_n^{(\beta)}) = 0$ if $\beta \neq \alpha$. This completes the proof of (2). q.e.d.

For $1 \leqq \alpha \leqq k$, let $I(F_\alpha)\colon {}^aS \longrightarrow H^*(F_\alpha; \mathbb{Q})$ be the ring homomorphism defined by setting $I(F_\alpha)(x_{i,j}) = a_{i,j}^{(\alpha)}$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq k_i)$, and $I((F_\alpha)_G)\colon S \longrightarrow H_G^*(F_\alpha; \mathbb{Q})$ a $\mathbb{Q}[t]$-algebra homomorphism defined by $I(F_\alpha)$ and ι_α. Choose $N_0 \geqq N$ and $g_{i,j} \in {}^aR$ $(1 \leqq i \leqq n,\ 1 < j \leqq k_i)$ such that

$$I((F_\alpha)_G)(J_\alpha(g_{i,j})) = t^{N_0 - j} p_\alpha{}^*(a_{i,j}^{(\alpha)}) \quad (1 \leqq \alpha \leqq k).$$

For $1 \leqq \alpha \leqq k$, let I_α be the ideal generated in ${}^aS$ by the coefficients of $J_\alpha(f_i)$ $(1 \leqq i \leqq m)$ and $J_\alpha(g_{i,j}) - x_{i,j} t^{N_0 - j}$

$(1 \leqq i \leqq n,\ 1 < j \leqq k_i)$ with respect to t, and set $q_\alpha = J_\alpha^{-1}(I_\alpha S)$.

Theorem 3.7. Let α be an integer with $1 \leqq \alpha \leqq k$. Then we have

(1) The kernel of the homomorphism $I(F_\alpha)$ coincides with I_α, and $I(F_\alpha)$ induces the isomorphism $i(F_\alpha)$: ${}^aS/I_\alpha \longrightarrow H^*(F_\alpha; \mathbb{Q})$.

(2) $(f_1, \cdots, f_m) = \cap_{\alpha=1}^k q_\alpha$ is the reduced primary decomposition, where $\sqrt{q_\alpha} = (x_1 - c_1^{(\alpha)} t^{k_1}, \cdots, x_n - c_n^{(\alpha)} t^{k_n})$.

(3) $g_{i,j}(1, c_1^{(\alpha)}, \cdots, c_n^{(\alpha)}) = 0 \quad (1 \leqq i \leqq n,\ 1 < j \leqq k_i)$.

Proof. By the definition, we have $I(F_\alpha)(I_\alpha) = 0$. Let $f \in \operatorname{Ker} I(F_\alpha)$ be a homogeneous element. By Lemma 3.6 (2), there is an integer $i(\beta)$ with $c_{i(\beta)}^{(\beta)} \neq c_{i(\beta)}^{(\alpha)}$ for each $\beta \neq \alpha$. Set

$$g = f(g_{i,j} t^{N_0(j-1)}) \cdot \Pi_{\beta \neq \alpha}(x_{i(\beta)} - c_{i(\beta)}^{(\beta)} t^{k_{i(\beta)}})^{N_1},$$

where N_1 is an integer such that $H^i(F; \mathbb{Q}) = 0$ for $i \geqq 2N_1$ and $g_{i,1} = x_i t^{N_0 - k_i} - c_i^{(\alpha)} t^{N_0} - \sum_{j=2}^{k_i} g_{i,j}$ $(1 \leqq i \leqq n)$. Then we have $j^*(I(X_G)(g)) = 0$, and hence $g \in (f_1, \cdots, f_m)$. The coefficient of the highest degree with respect to t in the polynomial $J_\alpha(g)$ is congruent to a multiple of f by some non-zero constant (mod I_α). This implies that $f \in I_\alpha$. This completes the proof of (1).

Since $\dim_{\mathbb{Q}} {}^aS/I_\alpha = \dim_{\mathbb{Q}} H^*(F_\alpha; \mathbb{Q}) < \infty$, we have $\sqrt{I_\alpha} = (x_{i,j})$, and hence I_α is a primary ideal. It follows that $I_\alpha S$ is also a primary ideal and $\sqrt{I_\alpha S} = (x_{i,j})$. By the definition, q_α is a primary ideal with the radical

$(x_1-c_1^{(\alpha)}t^{n_1},\cdots,x_n-c_n^{(\alpha)}t^{n_n})$. Since j^* is a monomorphism, we have $(f_1,\cdots,f_m) = \cap_{\alpha=1}^k q_\alpha$. This completes the proof of (2).

Since $g_{i,j}(1,c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) \in \mathbb{Q}$ is equal to the coefficient of t^{N_0} of the polynomial $J_\alpha(g_{i,j}) - x_{i,j}t^{N_0-j}$ and $I_\alpha \neq {}^aS$, we have $g_{i,j}(1,c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) = 0$. q.e.d.

By Lemma 2.3, it is easy to see that we can assume $N_0 = N$. According to [5] and Lemma 2,3, we have the following theorem.

Theorem 3.8 (V. Puppe [5]). Let $f_i \in R$ and $\varphi_i \in {}^aR$ $(1 \leqq i \leqq m)$ be homogeneous elements that satisfy (I) and (II). Let $\{(c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) | 1 \leqq \alpha \leqq k\}$ be a set of rational zero points of the ideal ${}^a(f_1,\cdots,f_m) \subset {}^aR$. For $1 \leqq i \leqq n$ and $1 < j \leqq k_i$, $g_{i,j} \in R$ denotes either 0 or a homogeneous element of degree $2N$ and $g_{i,j}(1,c_1^{(\alpha)},\cdots,c_n^{(\alpha)}) = 0$ $(1 \leqq \alpha \leqq k)$. Set I_α as in the section 2 $(1 \leqq \alpha \leqq k)$. If one of the properties i)-vi) of (7) of Lemma 2.3 is satisfied, then there is a finite G-CW-pair (X, F) and $\mathbb{Q}[t]$-algebra isomorphisms

$$i(X_G): R/(f_1,\cdots,f_m) \longrightarrow H_G^*(X;\mathbb{Q})$$

and $i((F_\alpha)_G): S/I_\alpha S \longrightarrow H_G^*(F_\alpha;\mathbb{Q})$ $(1 \leqq \alpha \leqq k)$, such that $F = X^G$ with connected components $F_1, \cdots, F_k$ and the following diagram commutes:

$$\begin{array}{ccc} R/(f_1,\cdots,f_m) & \xrightarrow{\quad j \quad} & \oplus_{\alpha=1}^k S/I_\alpha S \\ \Big\downarrow i(X_G) & & \Big\downarrow \oplus_{\alpha=1}^k i((F_\alpha)_G) \\ H_G^*(X;\mathbb{Q}) & \xrightarrow{\quad j^* \quad} & \oplus_{\alpha=1}^k H_G^*(F_\alpha;\mathbb{Q}). \end{array}$$

4. Applications

Applying Theorems 3.7 and 3.8 to corresponding cases, we obtain the following theorems.

Theorem 4.1. Let $X \sim_{\mathbb{Q}} S^{2m} \times S^{2n}$ be a finite G-CW-complex, $1 \leqq m \leqq n$. Then one of the following possibilities must occur:

(1) $F \sim_{\mathbb{Q}} S^{2q} \times S^{2r}$, $m \geqq q$, $n \geqq r$.

(2) $F \sim_{\mathbb{Q}} P^3(2q)$, $1 \leqq q \leqq n/2 < m$ or $1 \leqq q \leqq m/2 \leqq n/4$.

(3) $F \sim_{\mathbb{Q}}$ (point + $P^2(2q)$), $1 \leqq q \leqq n/2 < m$.

(4) $F \sim_{\mathbb{Q}} S^{2q} + S^{2r}$, $n \geqq q$, $n \geqq r$.

Conversely, each type of (1)-(4) can be realized by the fixed point set of a G-CW-complex.

Theorem 4.2. Let $X \sim_{\mathbb{Q}} HP(2) \# CP(4)$ be a finite G-CW-complex. Then one of the following possibilities must occur:

(0) $F \sim_{\mathbb{Q}} X$.

(1) $F \sim_{\mathbb{Q}} CP(2) \# CP(2) + S^2$.

(2) $F \sim_{\mathbb{Q}} F_1$ + 2 points,

where $H^*(F;\mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1x_2, x_2{}^2-\alpha x_1{}^2)$, $0 \neq \alpha \in \mathbb{Q}$ and $\deg x_1 = \deg x_2 = 2$.

(3) $F \sim_{\mathbb{Q}} CP(3)$ + 2 points.

(4) $F \sim_{\mathbb{Q}} CP(2) + CP(2)$.

(5) $F \sim_{\mathbb{Q}} CP(2) + S^{2k}$ + point $(k \leqq 2)$.

(6) $F \sim_{\mathbb{Q}} S^{2k} + S^{2m}$ + 2 points $(k \leqq 2,\ m \leqq 1)$.

Conversely, each type of (1)-(6) can be realized by the fixed point set of a G-CW-complex.

Theorem 4.3. Let $X \sim_{\mathbb{Q}} HP(2) \# (-CP(4))$ be a finite G-CW-complex. Then one of the following possibilities must occur:

(0) $F \sim_{\mathbb{Q}} X$.

(1) $F \sim_{\mathbb{Q}} S^2 \times S^2 + S^2$.

(2) $F \sim_{\mathbb{Q}} F_1 + 2$ points,

where $H^*(F;\mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1x_2, x_2{}^2+\alpha x_1{}^2)$, $0 \neq \alpha \in \mathbb{Q}$ and $\deg x_1 = \deg x_2 = 2$.

(3) $F \sim_{\mathbb{Q}} CP(3) + S^{2k}$ $(k \leqq 2)$.

(4) $F \sim_{\mathbb{Q}} CP(2) + CP(2)$.

(5) $F \sim_{\mathbb{Q}} CP(2) + S^{2k} +$ point $(k \leqq 2)$.

(6) $F \sim_{\mathbb{Q}} S^{2k} + S^{2m} + S^{2n}$ $(k \leqq 2,\ m \leqq 1,\ n \leqq 1)$.

Conversely, each type of (1)-(6) can be realized by the fixed point set of a G-CW-complex.

5. Construction of S^1-CW-complexes

Finally we construct some G-CW-complexes which give examples in the Theorems 4.2 and 4.3. Set $S^{11} = \{(u_1+jv_1, u_2+jv_2, u_3+jv_3) \in (\mathbb{C} \oplus j\mathbb{C})^3 \mid \sum_{i=1}^3 (|u_i|^2+|v_i|^2) = 1\}$ and $S^3 = \{x+jy \in \mathbb{C} \oplus j\mathbb{C} \mid |x|^2+|y|^2 = 1\}$. Then $HP(2)$ is defined as the orbit space S^{11}/S^3, where the S^3-action on S^{11} is defined by

$$\begin{aligned}&(u_1+jv_1, u_2+jv_2, u_3+jv_3)\cdot(x+jy)\\ &= (u_1x-\bar{v}_1y+j(v_1x+\bar{u}_1y), u_2x-\bar{v}_2y+j(v_2x+\bar{u}_2y), u_3x-\bar{v}_3y+j(v_3x+\bar{u}_3y))\end{aligned}$$

for every $(u_1+jv_1, u_2+jv_2, u_3+jv_3, x+jy) \in S^{11} \times S^3$. For each $(c_1, c_2, c_3) \in \mathbb{Z}^3$, $\Psi(c_1, c_2, c_3)\colon G \times HP(2) \longrightarrow HP(2)$ denotes the G-action on $HP(2)$ defined by

$$\Psi(c_1,c_2,c_3)(z,[u_1+jv_1,u_2+jv_2,u_3+jv_3])$$
$$= [z^{c_1}u_1+jz^{-c_1}v_1, z^{c_2}u_2+jz^{-c_2}v_2, z^{c_3}u_3+jz^{-c_3}v_3]$$

for every $(z,[u_1+jv_1,u_2+jv_2,u_3+jv_3]) \in G \times HP(2)$. Set

$$D_1^8 = \{[u_1+jv_1,u_2+jv_2,u_3+jv_3] \in HP(2) \mid |u_1|^2+|v_1|^2 > 1/2\},$$
$$S_1^7 = \{[u_1+jv_1,u_2+jv_2,u_3+jv_3] \in HP(2) \mid |u_1|^2+|v_1|^2 = 1/2\}$$

and $S_q^7 = \{(u_1+jv_1,u_2+jv_2) \in (\mathbb{C} \oplus j\mathbb{C})^2 \mid \sum_{i=1}^2(|u_i|^2+|v_i|^2) = 1\}$.
Let $h_1\colon S_1^7 \longrightarrow S_q^7$ be the homeomorphism defined by

$$h_1([u_1+jv_1,u_2+jv_2,u_3+jv_3])$$
$$= (1/(u_1\bar{u}_1+v_1\bar{v}_1))(u_2\bar{u}_1+\bar{v}_2v_1+j(v_2\bar{u}_1-\bar{u}_2v_1), u_3\bar{u}_1+\bar{v}_3v_1+j(v_3\bar{u}_1-\bar{u}_3v_1))$$

for every $[u_1+jv_1,u_2+jv_2,u_3+jv_3] \in S_1^7$. Set

$$\Sigma^7 = \{(r_1z_1,r_2z_2,r_3z_3,r_4z_4) \mid \sum_{j=1}^4 r_j = 1,\ r_j \geq 0,\ z_j \in S^1\}.$$

Let $f_q\colon S_q^7 \longrightarrow \Sigma^7$ be the homeomorphism defined by

$$f_q(u_1+jv_1,u_2+jv_2) = (u_1/s,v_1/s,u_2/s,v_2/s)$$

for every $(u_1+jv_1,u_2+jv_2) \in S_q^7$ with $s = \sum_{j=1}^2(|u_j|+|v_j|)$. Set

$$S^9 = \{(w_0,w_1,w_2,w_3,w_4) \in \mathbb{C}^5 \mid \sum_{i=0}^4 |w_i|^2 = 1\},$$

and $S^1 = \{z \in \mathbb{C} \mid |z|^2 = 1\}$. Then $CP(4)$ is defined as the orbit space S^9/S^1, where the S^1-action on S^9 is defined by

$$z\cdot(w_0,w_1,w_2,w_3,w_4) = (zw_0,zw_1,zw_2,zw_3,zw_4)$$

for every $(z,w_0,w_1,w_2,w_3,w_4) \in S^1 \times S^9$. For each $(a_0,a_1,a_2,a_3,a_4) \in \mathbb{Z}^5$, $\Phi(a_0,a_1,a_2,a_3,a_4)\colon G \times CP(4) \longrightarrow CP(4)$ denotes the G-action on $CP(4)$ defined by

$$\Phi(a_0,a_1,a_2,a_3,a_4)(z,[w_0,w_1,w_2,w_3,w_4])$$
$$= [z^{a_0}w_0, z^{a_1}w_1, z^{a_2}w_2, z^{a_3}w_3, z^{a_4}w_4]$$

for every $(z,[w_0,w_1,w_2,w_3,w_4]) \in G \times CP(4)$. Set

$$D_2^8 = \{[w_0,w_1,w_2,w_3,w_4] \in CP(4) \mid |w_0|^2 > 1/2\},$$
$$S_2^7 = \{[w_0,w_1,w_2,w_3,w_4] \in CP(4) \mid |w_0|^2 = 1/2\}$$

and $S_c^7 = \{(w_1,w_2,w_3,w_4) \in \mathbb{C}^4 | \Sigma_{i=1}^4 |w_i|^2 = 1\}$. Let $h_2\colon S_2^7 \longrightarrow S_c^7$ be the homeomorphism defined by

$$h_2([w_0,w_1,w_2,w_3,w_4]) = (w_1/w_0,w_2/w_0,w_3/w_0,w_4/w_0)$$

for every $[w_0,w_1,w_2,w_3,w_4] \in S_2^7$. Let $f_c\colon S_c^7 \longrightarrow \Sigma^7$ be the homeomorphism defined by

$$f_c(w_1,w_2,w_3,w_4) = (w_1/s,w_2/s,w_3/s,w_4/s)$$

for every $(w_1,w_2,w_3,w_4) \in S_c^7$ with $s = \Sigma_{j=1}^4 |w_j|$. For each $(d_1,d_2,d_3,d_4) = \mathbb{Z}^4$, let $f(d_1,d_2,d_3,d_4)\colon \Sigma^7 \longrightarrow \Sigma^7$ be the map defined by

$$f(d_1,d_2,d_3,d_4)((r_1z_1,r_2z_2,r_3z_3,r_4z_4)) = (r_1z_1^{d_1},r_2z_2^{d_2},r_3z_3^{d_3},r_4z_4^{d_4})$$

for every $(r_1z_1,r_2z_2,r_3z_3,r_4z_4) \in \Sigma^7$. Now we set

$$X = X(d_1,d_2,d_3,d_4;b_1,b_2,b_3,b_4) = \Sigma^7 \cup_{f_q(d_1,d_2,d_3,d_4)} (HP(2)\setminus D_1^8) \cup_{f_c(b_1,b_2,b_3,b_4)} (CP(4)\setminus D_2^8),$$

where $f_q(d_1,d_2,d_3,d_4)$ (resp. $f_c(b_1,b_2,b_3,b_4)$) is the composition

$$f(d_1,d_2,d_3,d_4)\circ f_q\circ h_1\colon S_1^7 \longrightarrow \Sigma^7$$

(resp. $f(b_1,b_2,b_3,b_4)\circ f_c\circ h_2\colon S_2^7 \longrightarrow \Sigma^7$). Suppose $d_1(c_2-c_1) = b_1(a_1-a_0)$, $d_2(-c_2-c_1) = b_2(a_2-a_0)$, $d_3(c_3-c_1) = b_3(a_3-a_0)$ and $d_4(-c_3-c_1) = b_4(a_4-a_0)$. Then X has the G-action compatible with $\Psi(c_1,c_2,c_3)|(HP(2)\setminus D_1^8)$ and $\Phi(a_0,a_1,a_2,a_3,a_4)|(CP(4)\setminus D_2^8)$; that is, X has a G-CW-complex structure. It is easy to see that we have

$$H^*(X;\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2]/(x_1x_2,d_1d_2d_3d_4x_2^2+b_1b_2b_3b_4x_1^4,x_2^3,x_1^5).$$

If $b_1b_2b_3b_4d_1d_2d_3d_4 = -n^2$ (resp. $b_1b_2b_3b_4d_1d_2d_3d_4 = n^2$) for some positive integer n, then we have $X \sim_{\mathbb{Q}} HP(2) \# CP(4)$

(resp. $X \sim_{\mathbb{Q}} HP(2) \# (-CP(4))$. Thus we obtain following examples.

[(4.2)(1)] If $c_1 = c_2 = c_3 = c \neq 0$, $a_1 = a_3 = a_0$, $a_2 = a_4 = a_0 - 2c$, $d_i = 1$ $(1 \leqq i \leqq 4)$, $-b_1 = b_2 = b_3 = b_4 = 1$, then we have

$$X = HP(2) \# CP(4) \quad \text{and} \quad X^G = CP(2) \# CP(2) + S^2.$$

[(4.2)(2)] If $c_1 = c_2 = c_3 = c \neq 0$, $a_1 = a_3 = a_2 + 2c = a_4 - a = a_0$ $(a(a+2c) \neq 0)$, $d_i = b_i = 1$ $(i = 1, 2)$, $d_3 = -b_4 = 2c$ and $d_4 = b_3 = a$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# CP(4) \quad \text{and} \quad X^G = F_1 + S^0,$$

where $H^*(F_1; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1x_2, 2cx_2{}^2 + ax_1{}^2)$ and $\deg x_1 = \deg x_2 = 2$.

[(4.2)(3)] If $c_1 = 5c \neq 0$, $c_2 = 4c$, $c_3 = 13c$, $a_i = a_0 - 6c$ $(1 \leqq i \leqq 4)$, $d_i = 6$ $(1 \leqq i \leqq 4)$, $b_1 = 1$, $b_2 = 9$, $b_3 = -8$ and $b_4 = 18$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# CP(4) \quad \text{and} \quad X^G = CP(3) + S^0.$$

[(4.2)(5) k=2] If $c_1 = c \neq 0$, $c_2 = c_3 = 0$, $a_i = a_0 - c$ $(1 \leqq i \leqq 3)$, $a_4 = a_0 + c$, $d_i = b_i = 1$ $(1 \leqq i \leqq 3)$, $d_4 = 1$ and $b_4 = -1$, then we have

$$X = HP(2) \# CP(4) \quad \text{and} \quad X^G = CP(2) + S^4 + \text{point}.$$

[(4.2)(5) k=1] If $c_1 = c_2 = 0$, $c_3 = c \neq 0$, $a_1 = a_2 = a_0$, $a_3 = a_4 = a_0 - c$, $d_i = (-1)^i b_i = 1$ $(2 \leqq i \leqq 4)$ and $d_1 = b_1 = 1$, then we have

$$X = HP(2) \# CP(4) \quad \text{and} \quad X^G = CP(2) + S^2 + \text{point}.$$

[(4.2)(5) k=0] If $c_1 = c_2 = 0$, $c_3 = c \neq 0$, $a_1 = a_2 = a_0$, $a_3 = a_0 + c$, $a_4 = a_0 - c$, $d_i = b_i = 1$ $(2 \leqq i \leqq 4)$ and $d_1 = -b_1$

$= 1$, then we have

$$X = HP(2) \# CP(4) \quad \text{and} \quad X^G = CP(2) + 3 \text{ points}.$$

[(4.2)(6) $k=2$, $m=1$] If $c_1 = 2c \neq 0$, $c_2 = c_3 = 0$, $a_1-c = a_3-c = a_2+c = a_0$, $a_4 = a_0+16c$, $2d_i = (-1)^i b_i = 2$ $(1 \leqq i \leqq 3)$, $d_4 = 8$ and $b_4 = -1$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# CP(4) \quad \text{and} \quad X^G = S^4 + S^2 + S^0.$$

[(4.2)(6) $k=2$, $m=0$] If $c_1 = 2c \neq 0$, $c_2 = c_3 = 0$, $a_1+c = a_2-c = a_3+2c = a_4+8c = a_0$, $d_i = 1$ $(1 \leqq i \leqq 3)$, $d_4 = 4$, $b_1 = -b_2 = 2$ and $b_3 = b_4 = 1$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# CP(4) \quad \text{and} \quad X^G = S^4 + 4 \text{ points}.$$

[(4.2)(6) $k=m=1$] If $c_1 = c_2 = c \neq 0$, $c_3 = 0$, $a_1 = a_0$, $a_2 = a_0+2c$, $a_3 = a_4 = a_0-c$, $d_i = 1$ $(1 \leqq i \leqq 4)$ and $b_1 = -b_2 = b_3 = b_4 = 1$, then we have

$$X = HP(2) \# CP(4) \quad \text{and} \quad X^G = S^2 + S^2 + 2 \text{ points}.$$

[(4.2)(6) $k=1$, $m=0$] If $c_1 = c_2 = c \neq 0$, $c_3 = 0$, $a_1 = a_2+2c = a_3-c = a_4+c = a_0$, $d_i = 1$ $(1 \leqq i \leqq 4)$, $b_i = (-1)^i$ $(2 \leqq i \leqq 4)$ and $b_1 = 1$, then we have

$$X = HP(2) \# CP(4) \quad \text{and} \quad X^G = S^2 + 4 \text{ points}.$$

[(4.2)(6) $k=m=0$] If $c_1 = c \neq 0$, $c_2 = 2c$, $c_3 = 3c$, $a_1-c = a_2+3c = a_3+2c = a_4+4c = a_0$, $d_i = b_i = 1$ $(i \neq 3)$ and $d_3 = -b_3 = 1$, then we have

$$X = HP(2) \# CP(4) \quad \text{and} \quad X^G = 6 \text{ points}.$$

[(4.3)(1)] If $c_1 = c_2 = c_3 = c \neq 0$, $a_1 = a_3 = a_0$, $a_2 = a_4 = a_0-2c$, $d_i = b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4))$$

and $X^G = CP(2) \# (-CP(2)) + S^2 \sim_{\mathbb{Q}} S^2 \times S^2 + S^2$.

[(4.3)(2)] If $c_1 = c_2 = c_3 = c \neq 0$, $a_1 = a_3 = a_2+2c = a_4-a$

$= a_0$ $(a(a+2c) \neq 0)$, $d_i = b_i = 1$ $(i = 1, 2)$, $d_3 = -b_4 = 2c$ and $d_4 = -b_3 = a$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = F_1 + S^0,$$

where $H^*(F_1 : \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1x_2, 2cx_2{}^2 - ax_1{}^2)$ and $\deg x_1 = \deg x_2 = 2$.

[(4.3)(3) k=2] If $c_1 = c \neq 0$, $c_2 = c_3 = 0$, $a_i = a_0 - c$ $(1 \leqq i \leqq 4)$ and $d_i = b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = CP(3) + S^4.$$

[(4.3)(3) k=1] If $c_1 = 0$, $c_2 = c_3 = c \neq 0$, $a_i = a_0 - c$ $(1 \leqq i \leqq 4)$, $d_i = 1$ $(1 \leqq i \leqq 4)$ and $b_i = (-1)^i$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = CP(3) + S^2.$$

[(4.3)(3) k=0] If $c_1 = 0$, $c_2 = c \neq 0$, $c_3 = 4c$, $a_i = a_0 + 2c$ $(1 \leqq i \leqq 4)$, $d_i = 2$ $(1 \leqq i \leqq 4)$, $b_1 = -b_2 = 1$ and $b_3 = -b_4 = 4$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = CP(3) + S^0.$$

[(4.3)(5) k=2] If $c_1 = 2c \neq 0$, $c_2 = c_3 = 0$, $a_i = a_0 - c$ $(1 \leqq i \leqq 3)$, $a_4 = a_0 - 16c$, $2d_i = b_i = 2$ $(1 \leqq i \leqq 3)$, $d_4 = 16$ and $b_4 = 2$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = CP(2) + S^4 + \text{point}.$$

[(4.3)(5) k=1] If $c_1 = c_2 = 0$, $c_3 = c \neq 0$, $a_1 = a_2 = a_0$, $a_3 = a_4 = a_0 - c$ and $d_i = (-1)^i b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = CP(2) + S^2 + \text{point}.$$

[(4.3)(5) k=0] If $c_1 = c_2 = 0$, $c_3 = c \neq 0$, $a_1 = a_2 = a_0$, $a_3 = a_0 + c$, $a_4 = a_0 - c$ and $d_i = b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = CP(2) + 3 \text{ points}.$$

[(4.3)(6) k=2, m=n=1] If $c_1 = c \neq 0$, $c_2 = c_3 = 0$, $a_1 = a_3 = a_0+c$, $a_2 = a_4 = a_0-c$ and $d_i = (-1)^i b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = S^4 + S^2 + S^2.$$

[(4.3)(6) k=2, m=1, n=0] If $c_1 = 2c \neq 0$, $c_2 = c_3 = 0$, $a_1-c = a_3-c = a_2+c = a_0$, $a_4 = a_0-16c$, $2d_i = (-1)^i b_i = 2$ $(1 \leqq i \leqq 3)$, $d_4 = 8$ and $b_4 = 1$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = S^4 + S^2 + 2 \text{ points}.$$

[(4.3)(6) k=2, m=n=0] If $c_1 = 2c \neq 0$, $c_2 = c_3 = 0$, $a_1+c = a_2-c = a_3+2c = a_4-8c = a_0$, $d_i = 1$ $(1 \leqq i \leqq 3)$, $d_4 = 4$, $b_1 = -b_2 = 2$ and $b_3 = -b_4 = 1$, then we have

$$X \sim_{\mathbb{Q}} HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = S^4 + 4 \text{ points}.$$

[(4.3)(6) k=m=n=1] If $c_1 = 0$, $c_2 = c_3 = c \neq 0$, $a_1 = a_3 = a_0+c$, $a_2 = a_4 = a_0-c$ and $d_i = b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = S^2 + S^2 + S^2.$$

[(4.3)(6) k=m=1, n=0] If $c_1 = c_2 = c \neq 0$, $c_3 = 0$, $a_1 = a_0$, $a_2 = a_0+2c$, $a_3 = a_4 = a_0-c$, $d_i = 1$ $(1 \leqq i \leqq 4)$ and $-b_1 = -b_2 = b_3 = b_4 = 1$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = S^2 + S^2 + S^0.$$

[(4.3)(6) k=1, m=n=0] If $c_1 = c_2 = c \neq 0$, $c_3 = 0$, $a_0 = a_1 = a_2+2c = a_3-c = a_4+c$ and $d_i = (-1)^i b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = S^2 + 4 \text{ points}.$$

[(4.3)(6) k=m=n=0] If $c_1 = c \neq 0$, $c_2 = 2c$, $c_3 = 3c$, $a_1-c = a_2+3c = a_3-2c = a_4+4c = a_0$ and $d_i = b_i = 1$ $(1 \leqq i \leqq 4)$, then we have

$$X = HP(2) \# (-CP(4)) \quad \text{and} \quad X^G = 6 \text{ points}.$$

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